MTR08114 Robotics Homogeneous Transformation

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REMINDER 1: Properties of Rotation Matrices

SO(n) = Special Orthogonal Group of Order n

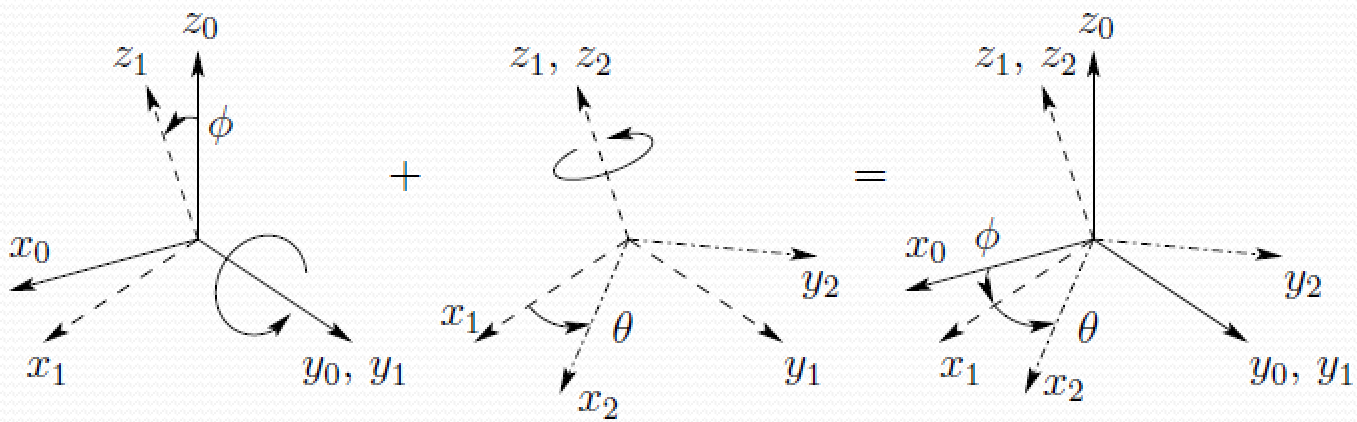
- $R \in SO(n)$
- $R^{-1} \in SO(n)$
- $R^{-1} = R^T$
- The columns (and therefore the rows) of R are mutually orthogonal
- Each column (and therefore each row) of R is a unit vector
- $\det R = 1$

$$\begin{aligned} \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} &= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^T \end{aligned}$$

REMINDER 2: Meanings of R

1. It represents a coordinate transformation relating the coordinates of a point p in two different frames.
2. It gives the orientation of a transformed coordinate frame with respect to a fixed coordinate frame.
3. It is an operator taking a vector and rotating it to a new vector in the same coordinate system.

REMINDER 3: Composition of Rotations

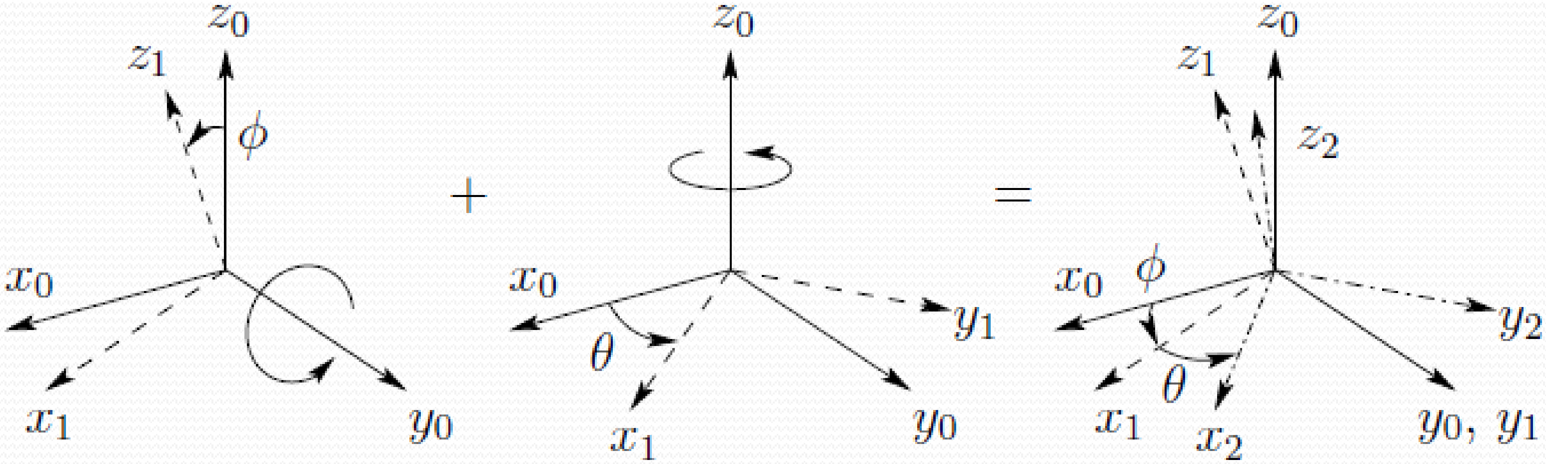


$$
\begin{aligned}
p^0 &= R_1^0 p^1 \\
p^1 &= R_2^1 p^2 \\
p^0 &= R_2^0 p^2 \\
p^0 &= R_1^0 R_2^1 p^2
\end{aligned}
$$

$$R_2^0 = R_1^0 R_2^1$$

REMINDER 4: Rotation in Fixed Frame

- Rotation around one axis in a frame followed by another rotation around another axis in the ORIGINAL frame



$$R_2^0 = R_1^0 \left[(R_1^0)^{-1} R R_1^0 \right] = R R_1^0$$

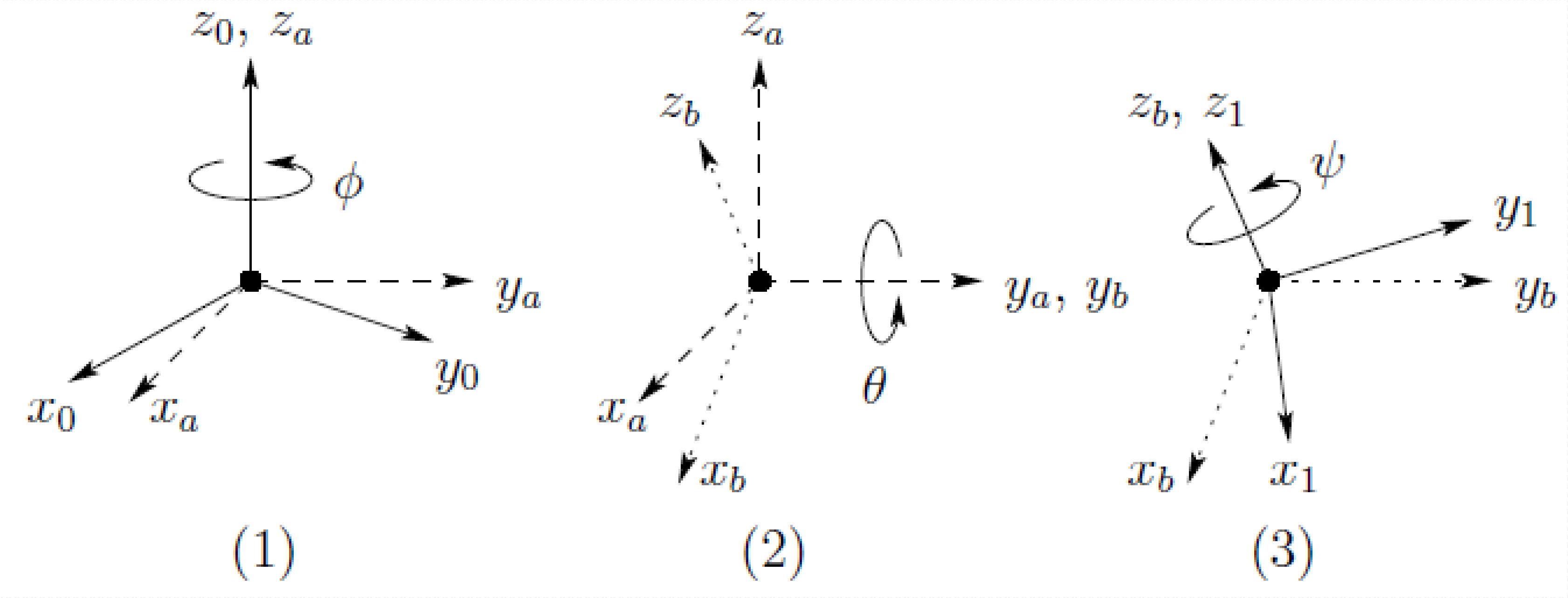
Rotation Representation

1. Direction Cosines
2. Eular Angle
3. Roll-Pitch-Yaw
4. Axis/Angle

- *Any 3D object possesses only 3 rotational degrees of freedom*

Eular Angles

- Φ, Θ, Ψ
- R_{ZYZ}



From Eular Angles to R

- Given Φ, Θ, Ψ
  - Find R

$$
\begin{aligned}
R_{ZYZ} &= R_{z,\phi} R_{y,\theta} R_{z,\psi} \\
&= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}
\end{aligned}
$$

From R to Eular Angles

- Given R
  - Find Φ, Θ, Ψ

$$\begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

- If r_{13} and r_{23} are not both zero
  - $s_\Theta \neq 0$, r_{31} and r_{32} are not both zero, $r_{33} \neq \pm 1$, $c_\Theta = r_{33}$, $s_\Theta = \pm\sqrt{1 - r_{33}}$

$$\begin{aligned} \theta &= \operatorname{atan2}\left(r_{33}, \sqrt{1 - r_{33}^2}\right) \\ \phi &= \operatorname{atan2}(r_{13}, r_{23}) \\ \psi &= \operatorname{atan2}(-r_{31}, r_{32}) \end{aligned}$$

$$\begin{aligned} \theta &= \operatorname{atan2}\left(r_{33}, -\sqrt{1 - r_{33}^2}\right) \\ \phi &= \operatorname{atan2}(-r_{13}, -r_{23}) \\ \psi &= \operatorname{atan2}(r_{31}, -r_{32}) \end{aligned}$$

Direction Cosines

- Principal axis of the new frame in the reference frame
- Vectors of the Rotation Matrix

From R to Eular Angles 2

- Given R
  - Find Φ, Θ, Ψ

$$\begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

- If r_{13} and r_{23} are both zeros
  - $s_\Theta=0$, $r_{31} = r_{32} = 0$, $r_{33} = \pm 1$, $c_\Theta = 1$, $\Theta=0$ *(Two rotations around Z)*

$$\begin{bmatrix} c_\phi c_\psi - s_\phi s_\psi & -c_\phi s_\psi - s_\phi c_\psi & 0 \\ s_\phi c_\psi + c_\phi s_\psi & -s_\phi s_\psi + c_\phi c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_{\phi+\psi} & -s_{\phi+\psi} & 0 \\ s_{\phi+\psi} & c_{\phi+\psi} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} \phi + \psi &= \text{atan2}(r_{11}, r_{21}) \\ &= \text{atan2}(r_{11}, -r_{12}) \end{aligned}$$

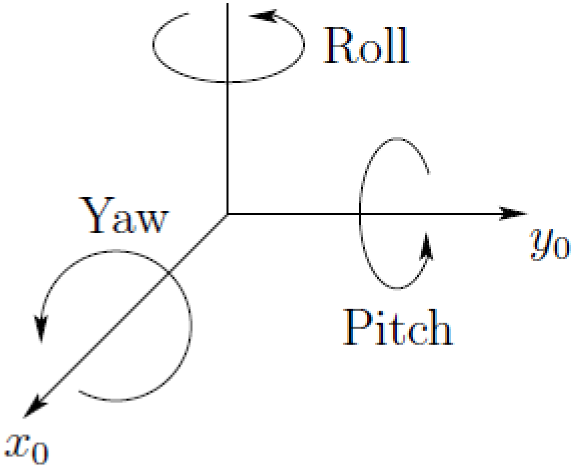
$$\begin{bmatrix} -c_{\phi-\psi} & -s_{\phi-\psi} & 0 \\ s_{\phi-\psi} & c_{\phi-\psi} & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\phi - \psi = \text{atan2}(-r_{11}, -r_{12})$$

Roll Pitch Yaw

$$
\begin{aligned}
R_{XYZ} &= R_{z,\phi}R_{y,\theta}R_{x,\psi} \\
&= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\psi & -s_\psi \\ 0 & s_\psi & c_\psi \end{bmatrix} \\
&= \begin{bmatrix} c_\phi c_\theta & -s_\phi c_\psi + c_\phi s_\theta s_\psi & s_\phi s_\psi + c_\phi s_\theta c_\psi \\ s_\phi c_\theta & c_\phi c_\psi + s_\phi s_\theta s_\psi & -c_\phi s_\psi + s_\phi s_\theta c_\psi \\ -s_\theta & c_\theta s_\psi & c_\theta c_\psi \end{bmatrix}
\end{aligned}
$$



- X→ Y → Z around fixed access
- Z → Y → X around current access

Rotation around a general axis

$$\sin\alpha = \frac{k_y}{\sqrt{k_x^2 + k_y^2}}$$

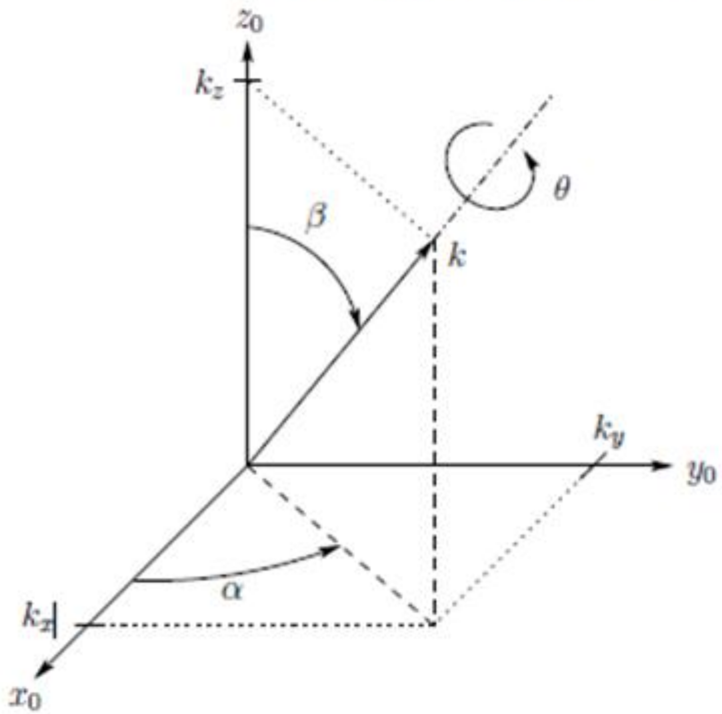
$$\cos\alpha = \frac{k_x}{\sqrt{k_x^2 + k_y^2}}$$

$$\sin\beta = \sqrt{k_x^2 + k_y^2}$$

$$\cos\beta = k_z$$

$$R_{k,\theta} = R_1^0 R_{z,\theta} R_1^{0^{-1}}$$

$$= R_{z,\alpha} R_{y,\beta} R_{z,\theta} R_{y,-\beta} R_{z,-\alpha}$$



$$R_{k,\theta} = \left[\begin{array}{c|c|c} k_x^2 v_\theta + c_\theta & k_x k_y v_\theta - k_z s_\theta & k_x k_z v_\theta + k_y s_\theta \\ k_x k_y v_\theta + k_z s_\theta & k_y^2 v_\theta + c_\theta & k_y k_z v_\theta - k_x s_\theta \\ k_x k_z v_\theta - k_y s_\theta & k_y k_z v_\theta + k_x s_\theta & k_z^2 v_\theta + c_\theta \end{array}\right]$$

From R to Axis/Angle representation

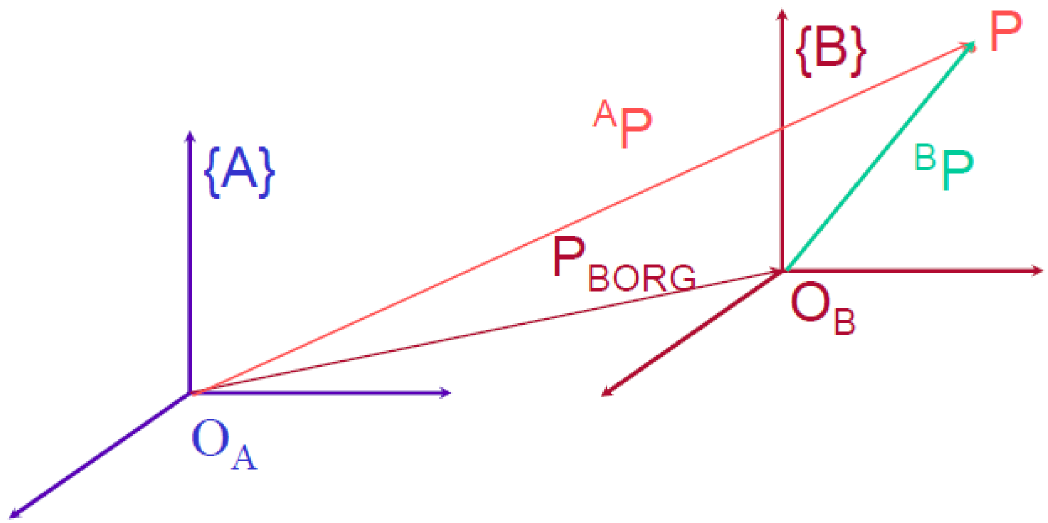
$$\begin{aligned} \theta &= \cos^{-1}\left(\frac{Tr(R)-1}{2}\right) \\ &= \cos^{-1}\left(\frac{r_{11}+r_{22}+r_{33}-1}{2}\right) \end{aligned}$$

$$k = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{bmatrix}$$

$$R_{k,\theta} = R_{-k,-\theta}$$

$$r = (r_x, r_y, r_z)^T = (\theta k_x, \theta k_y, \theta k_z)^T$$

Translation



changing the position description of a point P

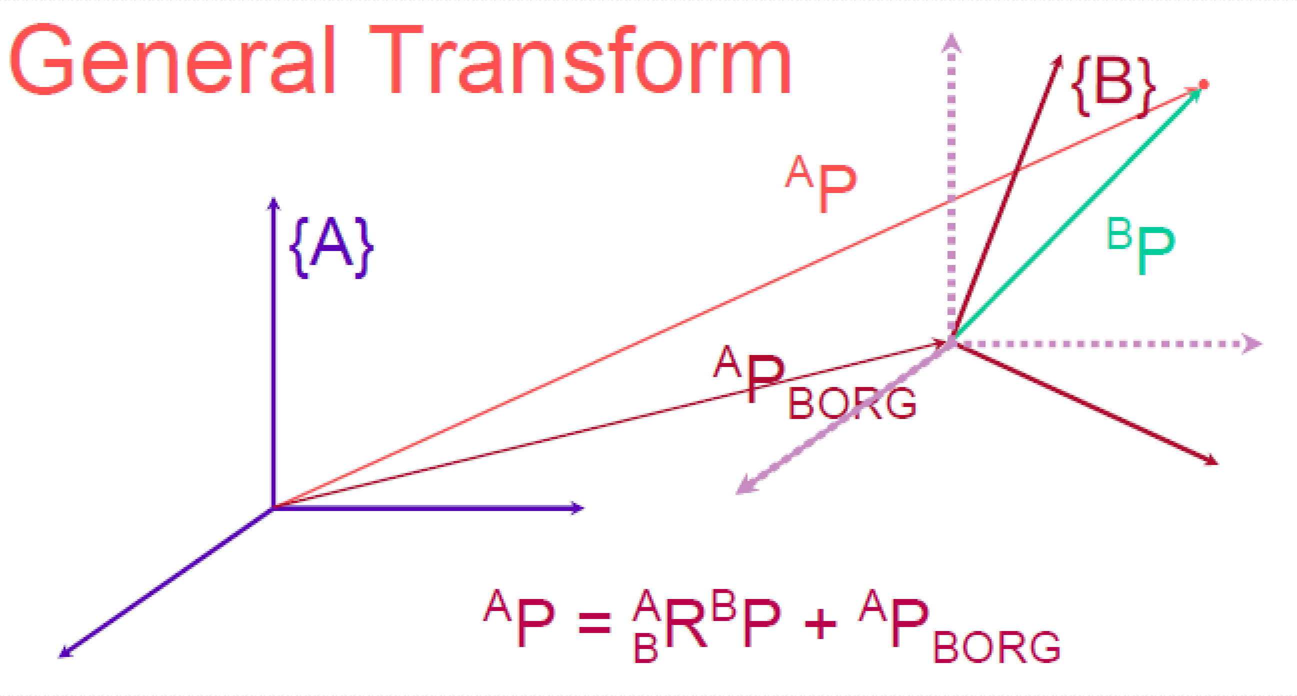
$\overrightarrow{O_BP} \Rightarrow \overrightarrow{O_AP}$ (Two different vectors)

$P_{BORG}: \quad P_{O_B} \Rightarrow P_{O_A}$

$$^AP_{O_A} = \overset{\overset{B}{\parallel}}{^AP_{O_B}} + {}^AP_{BORG}$$

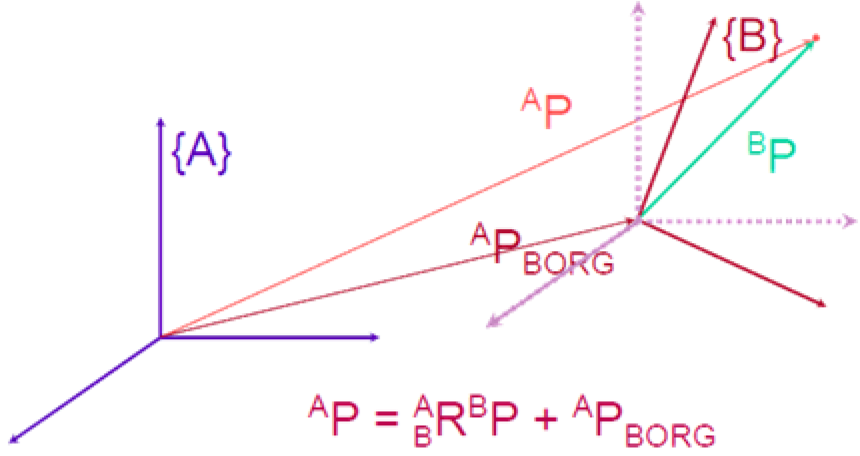
Rigid Transformation

- Rotation + Translation



Rigid Transformation 2

$$p^0 = R_1^0 p^1 + d_1^0$$



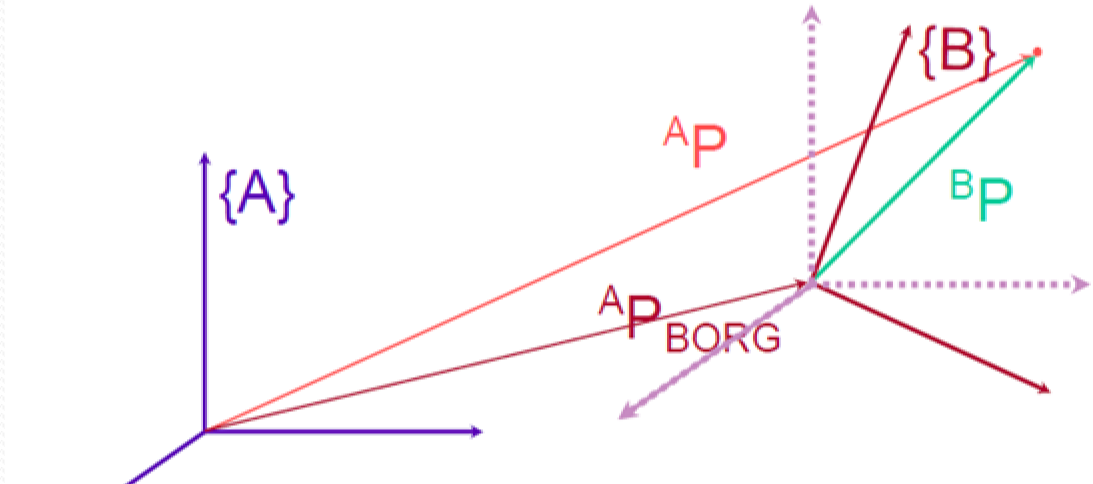
$$
\begin{aligned}
p^0 &= R_1^0 p^1 + d_1^0 \\
p^1 &= R_2^1 p^2 + d_2^1 \\
p^0 &= R_1^0 R_2^1 p^2 + R_1^0 d_2^1 + d_1^0 \\
p^0 &= R_2^0 p^2 + d_2^0
\end{aligned}
$$

$$
\begin{aligned}
R_2^0 &= R_1^0 R_2^1 \\
d_2^0 &= d_1^0 + R_1^0 d_2^1
\end{aligned}
$$

Homogeneous Transformation

$$\begin{bmatrix} {}^{A}P \\ \hline 1 \end{bmatrix} = \left[\begin{array}{c|c} {}^{A}_{B}R & {}^{A}P_{Borg} \\ \hline 0 \quad 0 \quad 0 & 1 \end{array}\right] \begin{bmatrix} {}^{B}P \\ \hline 1 \end{bmatrix}$$

$$\underset{(4x1)}{{}^{A}P} = \underset{(4x4)}{{}^{A}_{B}T} \; \underset{(4x1)}{{}^{B}P}$$



$$ {}^{A}P = {}^{A}_{B}R\,{}^{B}P + {}^{A}P_{BORG} $$

$$H = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}; R \in SO(3),\; d \in \mathbb{R}^3$$

Inverse Homogeneous

$$H^{-1} = \begin{bmatrix} R^T & -R^T d \\ 0 & 1 \end{bmatrix}$$

$$P^0 = \begin{bmatrix} p^0 \\ 1 \end{bmatrix}$$

$$P^1 = \begin{bmatrix} p^1 \\ 1 \end{bmatrix}$$

$$P^0 = H_1^0 P^1$$

Basic Homogeneous Trans.

$$\text{Trans}_{x,a} = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad Rot_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\alpha & -s_\alpha & 0 \\ 0 & s_\alpha & c_\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Trans}_{y,b} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad Rot_{y,\beta} = \begin{bmatrix} c_\beta & 0 & s_\beta & 0 \\ 0 & 1 & 0 & 0 \\ -s_\beta & 0 & c_\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Trans}_{z,c} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad Rot_{x,\gamma} = \begin{bmatrix} c_\gamma & -s_\gamma & 0 & 0 \\ s_\gamma & c_\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Anatomy of Homogeneous Trans.

Direction Cosines

Translation Vector

$$H_1^0 = \begin{bmatrix} n_x & s_x & a_x & d_x \\ n_y & s_y & a_y & d_y \\ n_z & s_x & a_z & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} n & s & a & d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Composition Rules

- Around current axis

$$H_2^0 = H_1^0 H$$

- Around fixed axis

$$H_2^0 = H H_1^0$$

Example

The homogeneous transformation matrix H *that represents a rotation by angle* α *about the current* x*-axis followed by a translation of* b *units along the current* x*-axis, followed by a translation of* d *units along the current* z*-axis, followed by a rotation by angle* θ *about the current* z*-axis, is given by*

$$
\begin{aligned}
H &= Rot_{x,\alpha} Trans_{x,b} Trans_{z,d} Rot_{z,\theta} \\
&= \begin{bmatrix} c_\theta & -s_\theta & 0 & b \\ c_\alpha s_\theta & c_\alpha c_\theta & -s_\alpha & -d s_\alpha \\ s_\alpha s_\theta & s_\alpha c_\theta & c_\alpha & d c_\alpha \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

General Homogeneous Transformation

$$H = \left[\begin{array}{c|c} R_{3\times3} & d_{3\times1} \\ \hline f_{1\times3} & s_{1\times1} \end{array}\right] = \left[\begin{array}{c|c} Rotation & Translation \\ \hline perspective & scale\ factor \end{array}\right]$$