MTR08114 Robotics Jacobian

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REMINDER 1: Velocity Kinematics

- Relation between end effector's linear and angular velocities and joint velocities.
- This is defined by the Jacobian (one of the most important concepts in robot motion)
- Steps:
  - Understand velocity and its transfer with moving frames!!
  - Derive Jacobian
  - Understand singularities

REMIDNER 2: Angular Velocity: FIXED AXIS

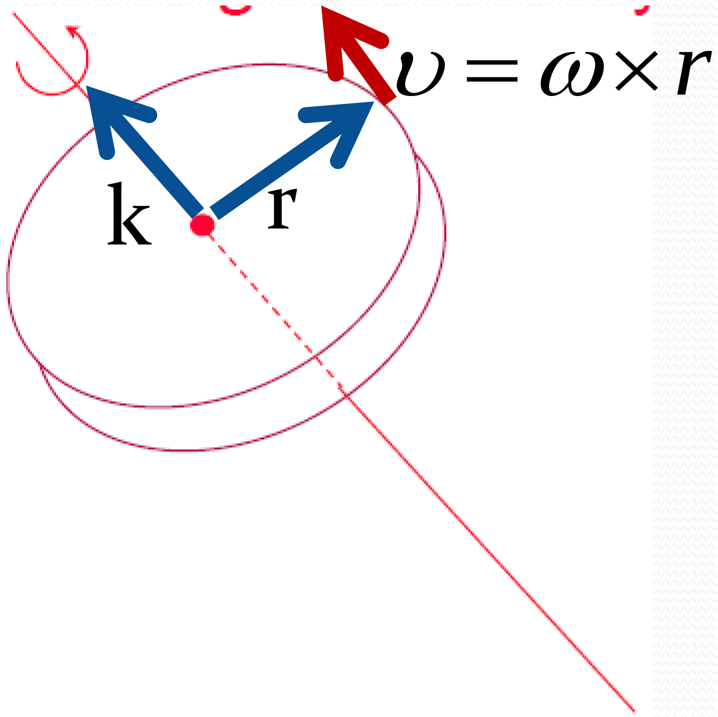
- Angular Velocity (describes a frame)

$$\omega = \dot{\theta}k$$

- Linear velocity (describes a point)

$$\upsilon = \omega \times r$$

- Angular velocity if fixed for the wh body
- Linear velocity depends on the distance between the point and the axis of rotation



- *How to represent angular velocity?*

REMINDER 3: Linear Velocity of a Point in a moving frame

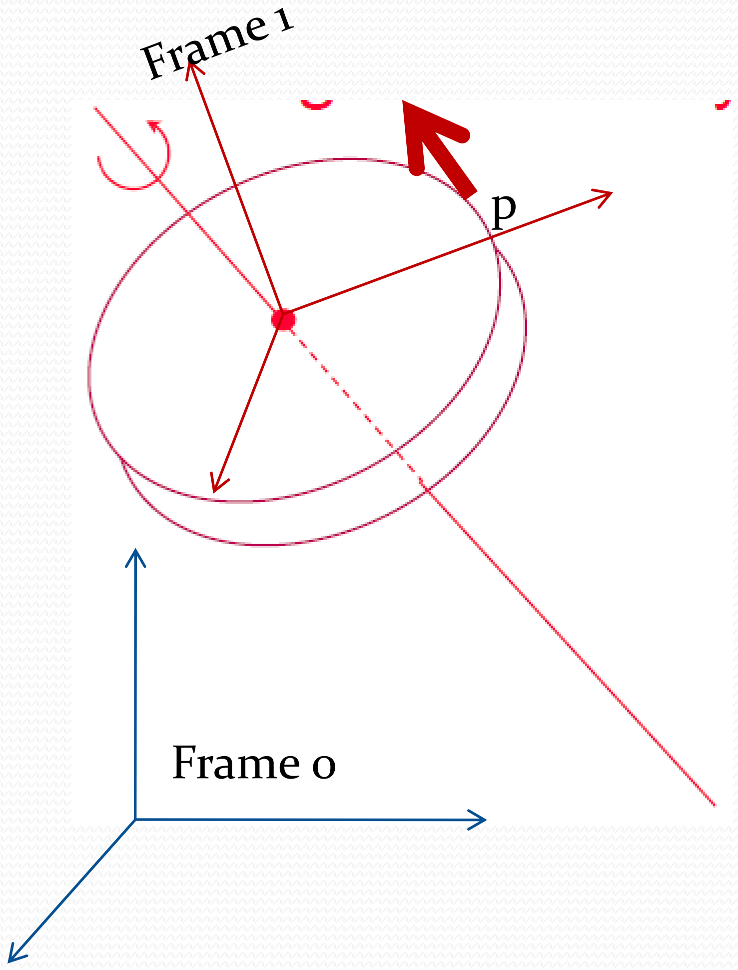
$$\upsilon = \omega \times p$$

- Translating And Rotating:

$$H_1^0(t) = \begin{bmatrix} R_1^0(t) & o_1^0(t) \\ 0 & 1 \end{bmatrix}$$

$$p^0 = Rp^1 + o$$

$$\begin{aligned} \dot{p}^0 &= \dot{R}p^1 + \dot{o} \\ &= S(\omega)Rp^1 + \dot{o} \\ &= \omega \times r + v \end{aligned}$$



What are we after?

- Given $T_n^0(q) = \begin{bmatrix} R_n^0(q) & o_n^0(q) \\ 0 & 1 \end{bmatrix}$
- Let $S(\omega_n^0) = \dot{R}_n^0 (R_n^0)^T$

  $v_n^0 = \dot{o}_n^0$

- Find $J = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix}$ where $\zeta = \begin{bmatrix} v_n^0 \\ \omega_n^0 \end{bmatrix} = J\dot{q} = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix} \dot{q}$

  Body Velocity (ζ)

  Jacobian (J)

The angular Jacobian J_ω

- Angular velocities are added as free vectors

$$\omega_2^0 = \omega_{0,1}^0 + R_1^0 \omega_{1,2}^1$$

- We can find ω_n by adding all ω_i's from the base to end effector

- Now $\omega = \dot{\theta} k$ where k is a unit vector in direction of rotation axis
- Using DH parameter's convention:

$$\omega_i^{i-1} = \dot{\theta}_i z_{i-1}^{i-1} = \dot{\theta}_i \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$$

J_ω (Prismatic Joint) $\omega_i^{i-1} = \dot{\theta}_i z_{i-1}^{i-1} = \dot{\theta}_i \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$

- Θ_i is constant

$$\omega_i^{i-1} = 0$$

J_ω (Revolute Joint)

- Θ_i is variable

$$\omega_i^{i-1} = \dot{\theta}_i \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$$

Angular Velocity of End Effector

- We know that:

$$\begin{aligned}\omega^0_{0,n} &= \omega^0_{0,1} + R^0_1\omega^1_{1,2} + R^0_2\omega^2_{2,3} + R^0_3\omega^3_{3,4} + \cdots + R^0_{n-1}\omega^{n-1}_{n-1,n} \\ &= \omega^0_{0,1} + \omega^0_{1,2} + \omega^0_{2,3} + \omega^0_{3,4} + \cdots + \omega^0_{n-1,n}\end{aligned}$$

- If all joints are revolute: $\omega^0_n = \sum_{i=0}^{n} \dot{\theta}_i R^0_{i-1} z^{i-1}_{i-1} = \sum_{i=0}^{n} \dot{\theta}_i z^0_{i-1}$

- If all of the m are prismatic $\omega^0_n = 0$

- In general $\omega^0_n = \sum_{i=0}^{n} \rho_i \dot{q}_i R^0_{i-1} z^{i-1}_{i-1} = \sum_{i=0}^{n} \rho_i \dot{q}_i z^0_{i-1}$

Now J_ω

$$\omega_n^0 = \sum_{i=0}^{n} \rho_i \dot{q}_i R_i^0 z_{i-1}^{i-1} = \sum_{i=0}^{n} \rho_i \dot{q}_i z_{i-1}^0$$

$$\omega_n^0 = \sum_{i=0}^{n} \left(\rho_i z_{i-1}^0 \right) \dot{q}_i = \sum_{i=0}^{n} J_{\omega i} \dot{q}_i$$

$$J_\omega = \left[\rho_i z_{i-1}^0 \right]_{i=1}^{n}$$

Linear Velocity and Jacobian

- By chain Rule of Differentiation:

$$\dot{o}_n^0 = \sum_{i=1}^{n} \frac{\partial o_n^0}{\partial q_i} \frac{dq_i}{dt} = \sum_{i=1}^{n} \frac{\partial o_n^0}{\partial q_i} \dot{q}_i$$

- By Jacobian definition $\dot{o}_n^0 = \sum_{i=1}^{n} J_{vi} \dot{q}_i$

$$J_v = \left[\frac{\partial o_n^0}{\partial q_i} \right]_{i=1}^{n}$$

What is the origin of frame *n* in 0

$$
\begin{aligned}
\begin{bmatrix} R_n^0 & o_n^0 \\ 0 & 1 \end{bmatrix} &= T_n^0 \\
&= T_{i-1}^0 T_i^{i-1} T_n^i \\
&= \begin{bmatrix} R_{i-1}^0 & o_{i-1}^0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_i^{i-1} & o_i^{i-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_n^i & o_n^i \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} R_n^0 & R_i^0 o_n^i + R_{i-1}^0 o_i^{i-1} + o_{i-1}^0 \\ 0 & 1 \end{bmatrix},
\end{aligned}
$$

$$
o_n^0 = R_i^0 o_n^i + R_{i-1}^0 o_i^{i-1} + o_{i-1}^0
$$

Now Differentiate

$$o_n^0 = R_i^0 o_n^i + R_{i-1}^0 o_i^{i-1} + o_{i-1}^0$$

$$\dot{o}_n^0 = \dot{R}_i^0 o_n^i + R_i^0 \dot{o}_n^i + \dot{R}_{i-1}^0 o_i^{i-1} + R_{i-1}^0 \dot{o}_i^{i-1} + \dot{o}_{i-1}^0$$

- If ONLY Joint *i* is moving

$$\dot{o}_n^0 = \dot{R}_i^0 o_n^i + 0 + \dot{R}_{i-1}^0 o_i^{i-1} + R_{i-1}^0 \dot{o}_i^{i-1} + 0$$

Prismatic Joint case

$$\dot{o}_n^0 = \dot{R}_i^0 o_n^i + R_i^0 \dot{o}_n^i + \dot{R}_{i-1}^0 o_i^{i-1} + R_{i-1}^0 \dot{o}_i^{i-1} + \dot{o}_{i-1}^0$$

- If ONLY Joint *i* is moving
  - $\dot{o}_n^0 = \dot{R}_i^0 o_n^i + 0 + \dot{R}_{i-1}^0 o_i^{i-1} + R_{i-1}^0 \dot{o}_i^{i-1} + 0$
  - R_{i-1}^0 and R_i^0 are constants
  - $\dot{o}_n^0 = R_{i-1}^0 \dot{o}_i^{i-1}$

Prismatic Joint

$$\dot{o}_n^0 = R_{i-1}^0 \dot{o}_i^{i-1} = R_{i-1}^0 \frac{\partial o_i^{i-1}}{\partial q_i} \dot{q}_i$$

$$\dot{o}_n^0 = \dot{q}_i R_{i-1}^0 \frac{\partial \begin{bmatrix} a_i c_i & a_i s_i & d_i \end{bmatrix}^T}{\partial d_i}$$

$$\therefore \dot{o}_n^0 = \dot{q}_i R_{i-1}^0 \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

$$\dot{o}_n^0 = \dot{q}_i z_{i-1}^0 \Rightarrow$$

$$\boxed{J_{vi} = z_{i-1}^0}$$

Revolute Joint case

$$\dot{o}_n^0 = \dot{R}_i^0 o_n^i + R_i^0 \dot{o}_n^i + \dot{R}_{i-1}^0 o_i^{i-1} + R_{i-1}^0 \dot{o}_i^{i-1} + \dot{o}_{i-1}^0$$

- If ONLY Joint *i* is moving
  - $\dot{o}_n^0 = \dot{R}_i^0 o_n^i + 0 + \dot{R}_{i-1}^0 o_i^{i-1} + R_{i-1}^0 \dot{o}_i^{i-1} + 0$
  - R_{i-1}^0 is constants
  - $\dot{o}_n^0 = \dot{R}_i^0 o_n^i + R_{i-1}^0 \dot{o}_i^{i-1}$

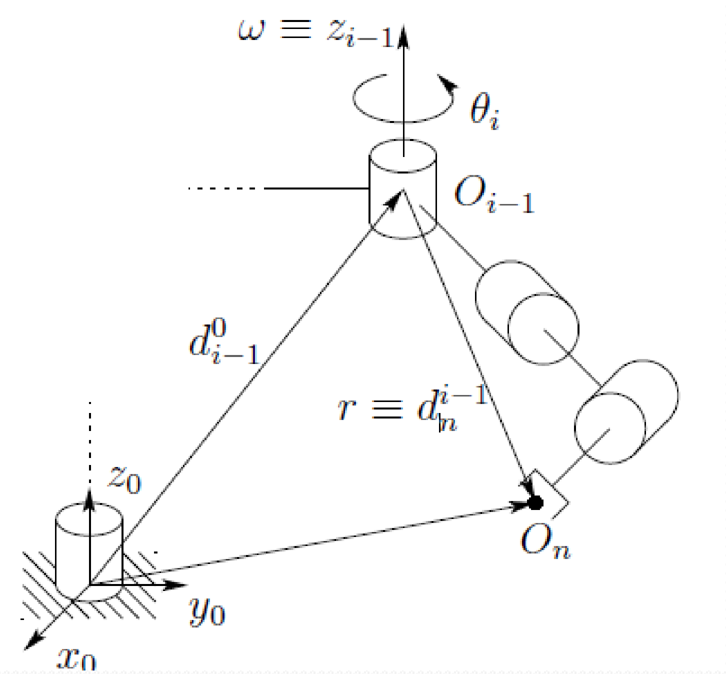
Revolute Joint

$$\begin{aligned}
\frac{\partial}{\partial \theta_i} o_n^0 &= \frac{\partial}{\partial \theta_i}\left[R_i^0 o_n^i + R_{i-1}^0 o_i^{i-1}\right] \\
&= \frac{\partial}{\partial \theta_i} R_i^0 o_n^i + R_{i-1}^0 \frac{\partial}{\partial \theta_i} o_i^{i-1} \\
&= \dot{\theta}_i S(z_{i-1}^0) R_i^0 o_n^i + \dot{\theta}_i S(z_{i-1}^0) R_{i-1}^0 o_i^{i-1} \\
&= \dot{\theta}_i S(z_{i-1}^0)\left[R_i^0 o_n^i + R_{i-1}^0 o_i^{i-1}\right] \\
&= \dot{\theta}_i S(z_{i-1}^0)(o_n^0 - o_{i-1}^0) \\
&= \dot{\theta}_i z_{i-1}^0 \times (o_n^0 - o_{i-1}^0)
\end{aligned}$$

More Details

$$
\begin{aligned}
R^0_{i-1}\frac{\partial}{\partial\theta_i}\begin{bmatrix} a_i c_i \\ a_i s_i \\ d_i \end{bmatrix} &= R^0_{i-1}\begin{bmatrix} -a_i s_i \\ a_i c_i \\ 0 \end{bmatrix}\dot{\theta}_i \\
&= R^0_{i-1}S(k\dot{\theta}_i)o_i^{i-1} \\
&= R^0_{i-1}S(k\dot{\theta}_i)\left(R^0_{i-1}\right)^T R^0_{i-1}o_i^{i-1} \\
&= S(R^0_{i-1}k\dot{\theta}_i)R^0_{i-1}o_i^{i-1} \\
&= \dot{\theta}_i S(z^0_{i-1})R^0_{i-1}o_i^{i-1}
\end{aligned}
$$

So for Revolute Joint



$$J_{v_i} = z_{i-1} \times (o_n - o_{i-1})$$

Putting It all Together

$$J_v = [J_{v_1} \cdots J_{v_n}]$$

$$J_{v_i} = \begin{cases} z_{i-1} \times (o_n - o_{i-1}) & \text{for revolute joint } i \\ z_{i-1} & \text{for prismatic joint } i \end{cases}$$

$$J_\omega = [J_{\omega_1} \cdots J_{\omega_n}]$$

$$J_{\omega_i} = \begin{cases} z_{i-1} & \text{for revolute joint } i \\ 0 & \text{for prismatic joint } i \end{cases}$$

The whole Jacobian (METHOD 1)

- Revolute Joint

$$J_i = \begin{bmatrix} z_{i-1} \times (o_n - o_{i-1}) \\ z_{i-1} \end{bmatrix}$$

- Prismatic Joint

$$J_i = \begin{bmatrix} z_{i-1} \\ 0 \end{bmatrix}$$

Where to get Them?

- Z from the third column of T
- O from the fourth column of T

$$T_i^0 = \begin{bmatrix} x_i^0 & y_i^0 & z_i^0 & o_i^0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The whole Jacobian (METHOD 2)*

- Revolute Joint

$$J_i = \begin{bmatrix} \dfrac{\partial o_n}{\partial q_i} \\ z_{i-1} \end{bmatrix}$$

- Prismatic Joint

$$J_i = \begin{bmatrix} z_{i-1} \\ 0 \end{bmatrix}$$

Example 1 (Planar RR)

$$o_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad o_1 = \begin{bmatrix} a_1 c_1 \\ a_1 s_1 \\ 0 \end{bmatrix} \quad o_2 = \begin{bmatrix} a_1 c_1 + a_2 c_{12} \\ a_1 s_1 + a_2 s_{12} \\ 0 \end{bmatrix}$$

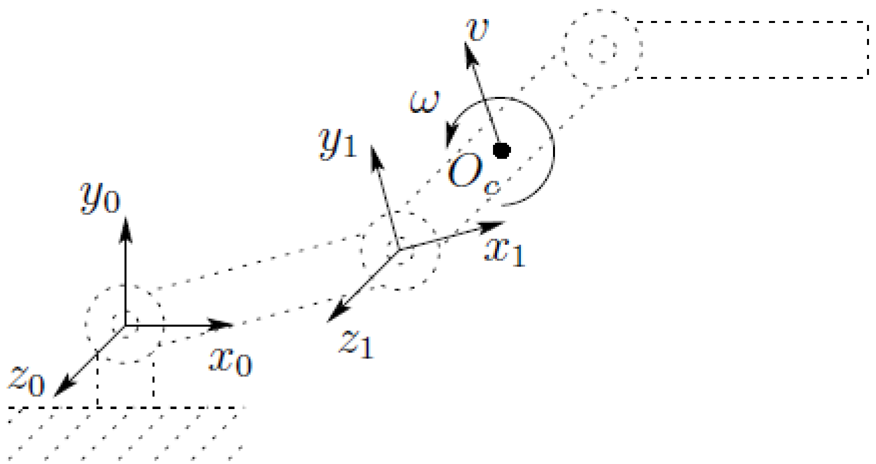
$$z_0 = z_1 \quad = \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$S(k) \quad = \quad \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$J(q) \quad = \quad \begin{bmatrix} z_0 \times (o_2 - o_0) & z_1 \times (o_2 - o_1) \\ z_0 & z_1 \end{bmatrix}$$

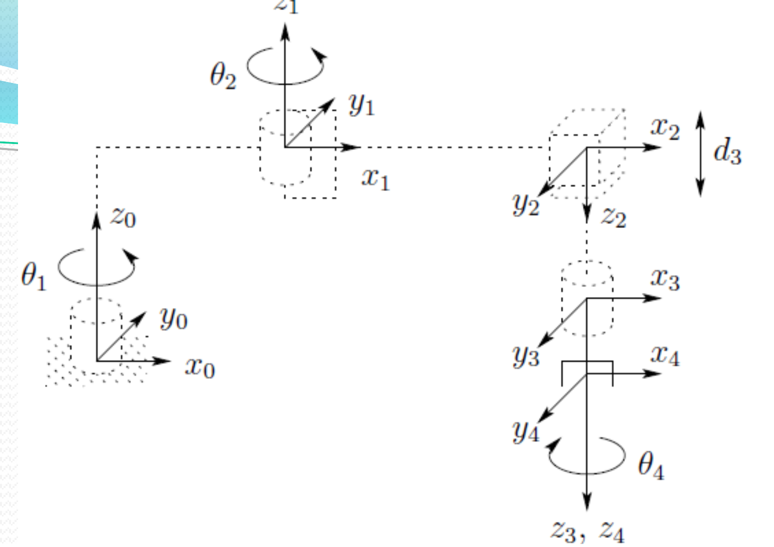
$$J \quad = \quad \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

Jacobian of Arbitrary Point



$$J(q) = \begin{bmatrix} z_0 \times (o_c - o_0) & z_1 \times (o_c - o_1) & 0 \\ z_0 & z_1 & 0 \end{bmatrix}$$

Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	$\theta^\star$
2	a_2	180	0	$\theta^\star$
3	0	0	$d^\star$	0
4	0	0	d_4	$\theta^\star$



Example 2: SCARA

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} c_2 & s_2 & 0 & a_2c_2 \\ s_2 & -c_2 & 0 & a_2s_2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} c_{12}c_4 + s_{12}s_4 & -c_{12}s_4 + s_{12}c_4 & 0 & a_1c_1 + a_2c_{12} \\ s_{12}c_4 - c_{12}s_4 & -s_{12}s_4 - c_{12}c_4 & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & -1 & -d_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$J = \begin{bmatrix} z_0 \times (o_4 - o_0) & z_1 \times (o_4 - o_1) & z_2 & 0 \\ z_0 & z_1 & 0 & z_3 \end{bmatrix}$$

$$o_1 = \begin{bmatrix} a_1c_1 \\ a_1s_1 \\ 0 \end{bmatrix} \qquad o_2 = \begin{bmatrix} a_1c_1 + a_2c_{12} \\ a_1s_1 + a_2s_{12} \\ 0 \end{bmatrix}$$

$$o_4 = \begin{bmatrix} a_1c_1 + a_2c_{12} \\ a_1s_2 + a_2s_{12} \\ d_3 - d_4 \end{bmatrix}$$

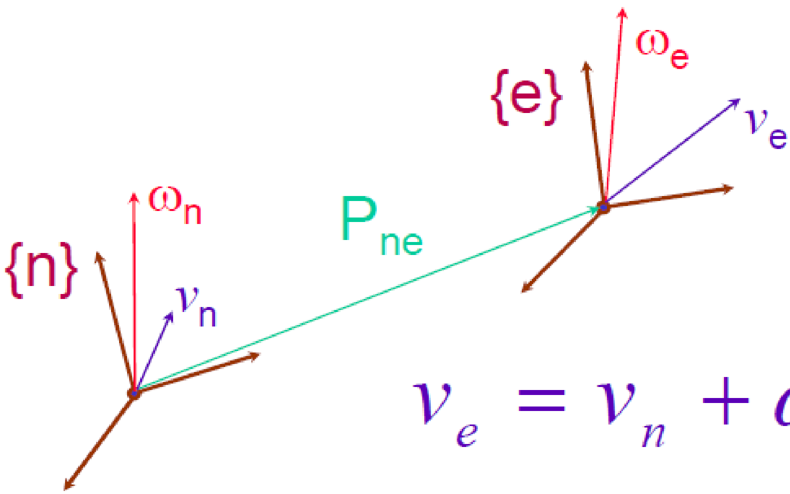
From A_1A_2

$$J = \begin{bmatrix} -a_1s_1 - a_2s_{12} & -a_2s_{12} & 0 & 0 \\ a_1c_1 + a_2c_{12} & a_2c_{12} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix}$$

Bit of Terminology

$$S(k) = \hat{k}$$

Jacobian at End Effector



$$v_e = v_n + \omega_n \times P_{ne}$$

$$\begin{cases} v_e = v_n - P_{ne} \times \omega_n \\ \omega_e = \omega_n \end{cases}$$

$$\begin{cases} v_e = v_n - P_{ne} \times \omega_n \\ \omega_e = \omega_n \end{cases}$$

$$\begin{pmatrix} v_e \\ \omega_e \end{pmatrix} = \left(\begin{array}{c|c} I & -\hat{P}_{ne} \\ \hline O & I \end{array}\right) \begin{pmatrix} v_n \\ \omega_n \end{pmatrix}$$

$$J_e \dot{q} = \left(\begin{array}{c|c} I & -\hat{P}_{ne} \\ \hline O & I \end{array}\right) J_n \dot{q}$$

$$J_e = \left(\begin{array}{c|c} I & -\hat{P}_{ne} \\ \hline O & I \end{array}\right) J_n$$

From Osama Khatib

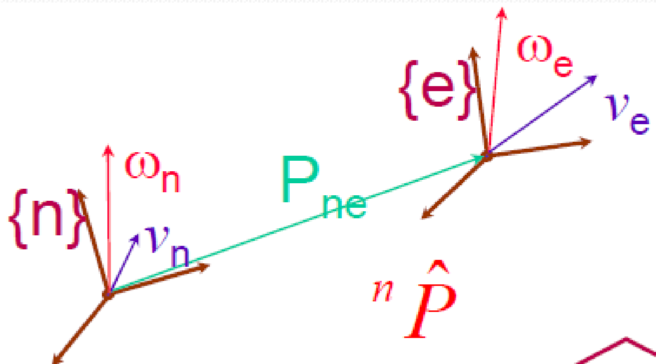
Jacobian in a different frame

$$ {}^{i}J = \begin{pmatrix} {}^{i}_{j}R & 0 \\ 0 & {}^{i}_{j}R \end{pmatrix} {}^{j}J $$

$$ \left| \det \quad {}^{i}J \right| = \left| \det \quad {}^{j}J \right| $$

From Osama Khatib

Cross Product in a different frame



$$^0J_e = \begin{pmatrix} I & -{}^0\hat{P}_{ne} \\ 0 & I \end{pmatrix} {}^0J_n$$

$$^n\hat{P}$$

$$^0\hat{P} \neq {}^0_nR\ {}^n\hat{P}; \quad \widehat{{}^0P} = \widehat{({}^0_nR.{}^nP)} \neq {}^0_nR.\widehat{{}^nP}$$

$$^0P\times{}^0\omega = {}^0_nR.({}^nP\times{}^n\omega)$$

$$^0\hat{P}.{}^0\omega = {}^0_nR.({}^n\hat{P}.{}^n\omega) = {}^0_nR.({}^n\hat{P}.{}^0_nR^T.{}^0\omega)$$

$$\boxed{^0\hat{P} = {}^0_nR\ {}^n\hat{P}\ {}^0_nR^T}$$

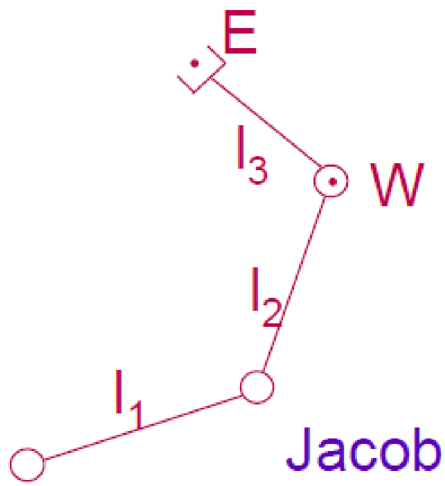
From Osama Khatib

Jacobian of End effector in first frame

$$ {}^{0}J_{e} = \begin{pmatrix} {}^{0}_{n}R & -{}^{0}_{n}R\,{}^{n}\hat{P}_{ne}\,{}^{0}_{n}R^{T} \\ 0 & {}^{0}_{n}R \end{pmatrix} {}^{n}J_{n} $$

From Osama Khatib

Example



Wrist Point

$$x = l_1 c_1 + l_2 c_{12}$$

$$y = l_1 s_1 + l_2 s_{12}$$

End-Effector Point

$$x = l_1 c_1 + l_2 c_{12} + l_3 c_{123}$$

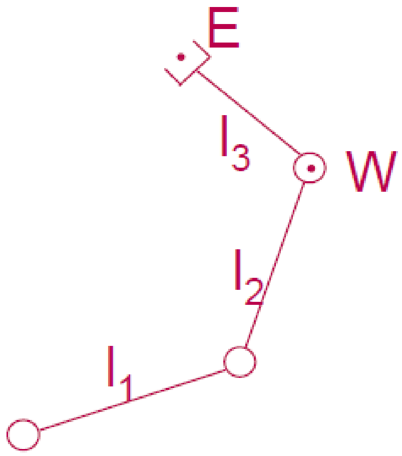
$$y = l_1 s_1 + l_2 s_{12} + l_3 s_{123}$$

Jacobian (W)

$$J_W = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} & 0 \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} ; \quad {}^0 J_E = \begin{pmatrix} I & -{}^0\hat{P}_{WE} \\ 0 & I \end{pmatrix} {}^0 J_W$$

From Osama Khatib

Example



Wrist Point

$$x = l_1 c_1 + l_2 c_{12}$$

$$y = l_1 s_1 + l_2 s_{12}$$

End-Effector Point

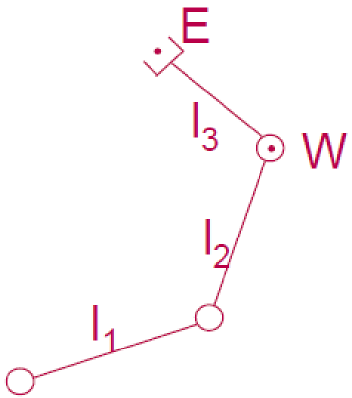
$$x = l_1 c_1 + l_2 c_{12} + l_3 c_{123}$$

$$y = l_1 s_1 + l_2 s_{12} + l_3 s_{123}$$

$$J_W = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} & 0 \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad {}^0J_E = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} - l_3 s_{123} & -l_2 s_{12} - l_3 s_{123} & -l_3 s_{123} \\ l_1 c_1 + l_2 c_{12} + l_3 c_{123} & l_2 c_{12} + l_3 c_{123} & l_3 c_{123} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

From Osama Khatib

Example



Wrist Point

$$x = l_1 c_1 + l_2 c_{12}$$

$$y = l_1 s_1 + l_2 s_{12}$$

End-Effector Point

$$x = l_1 c_1 + l_2 c_{12} + l_3 c_{123}$$

$$y = l_1 s_1 + l_2 s_{12} + l_3 s_{123}$$

$$J_W = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} & 0 \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$${}^0J_E = \begin{pmatrix} I & -{}^0\hat{P}_{WE} \\ 0 & I \end{pmatrix} {}^0J_W$$

$${}^0P_{WE} = \begin{bmatrix} l_3 c_{123} \\ l_3 s_{123} \\ 0 \end{bmatrix} \Rightarrow {}^0\hat{P}_{WE} = \begin{pmatrix} 0 & 0 & l_3 s_{123} \\ 0 & 0 & -l_3 c_{123} \\ -l_3 s_{123} & l_3 c_{123} & 0 \end{pmatrix}$$

From Osama Khatib

Singularity

- Configurations in which the rank of the Jacobian is less than 6

1. Singularities represent configurations from which certain directions of motion may be unattainable.

2. At singularities, bounded end-effector velocities may correspond to unbounded joint velocities.

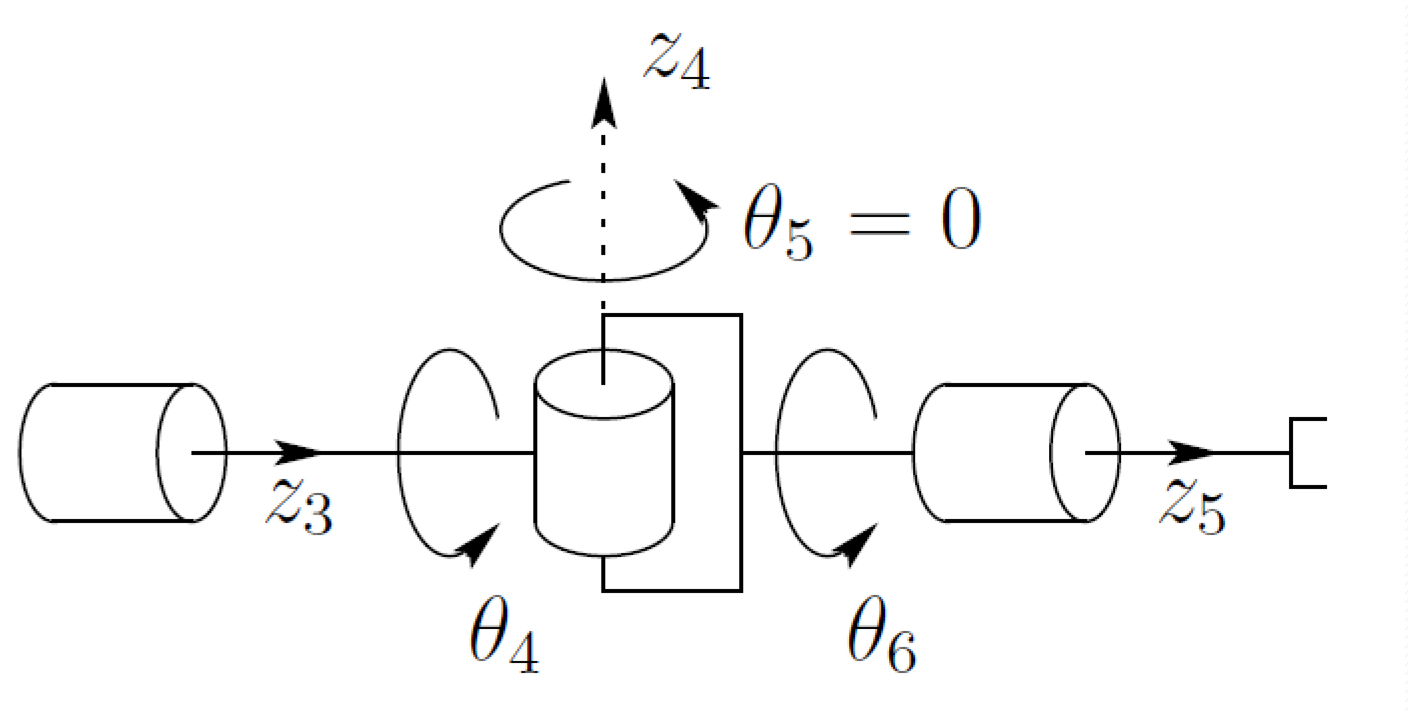
3. At singularities, bounded end-effector forces and torques may correspond to unbounded joint torques.

6. Near singularities there will not exist a unique solution to the inverse kinematics problem. In such cases there may be no solution or there may be infinitely many solutions.

How to find singularities?

1. Find the determinant of J and equalize it with zero
2. Find the QR decomposition of J and find the rank (does not give the singularity configuration)
3. For 6-DOF robots with wrist, find the determinant of the wrist alone and the arm alone. All singularities found are robot singularities.

Wrist Singularities

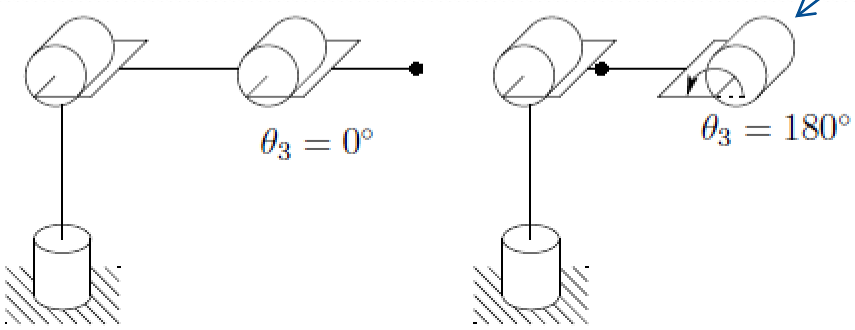


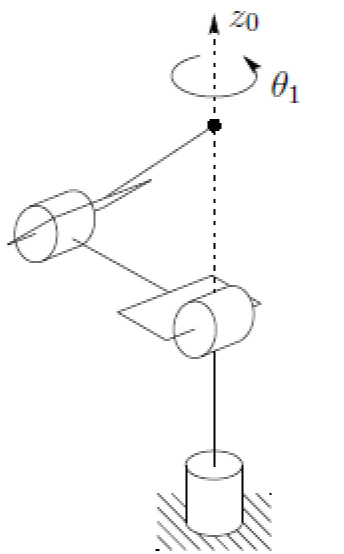
- Whenever z_3 and z_5 are aligned
- Prove it

Elbow Singularity

$$J_{11} = \begin{bmatrix} -a_2 s_1 c_2 - a_3 s_1 c_{23} & -a_2 s_2 c_1 - a_3 s_{23} c_1 & -a_3 c_1 s_{23} \\ a_2 c_1 c_2 + a_3 c_1 c_{23} & -a_2 s_1 s_2 - a_3 s_1 s_{23} & -a_3 s_1 s_{23} \\ 0 & a_2 c_2 + a_3 c_{23} & a_3 c_{23} \end{bmatrix}$$

$$\det J_{11} = a_2 a_3 s_3 (a_2 c_2 + a_3 c_{23}).$$





Resolved Motion Rate Control (Whitney 1972)

$$\delta x = J(\theta)\delta\theta$$

Outside singularities

$$\delta\theta = J^{-1}(\theta)\delta x$$

Arm at Configuration θ

$$x = f(\theta)$$

$$\delta x = x_d - x$$

$$\delta\theta = J^{-1}\delta x$$

$$\theta^{+} = \theta + \delta\theta$$

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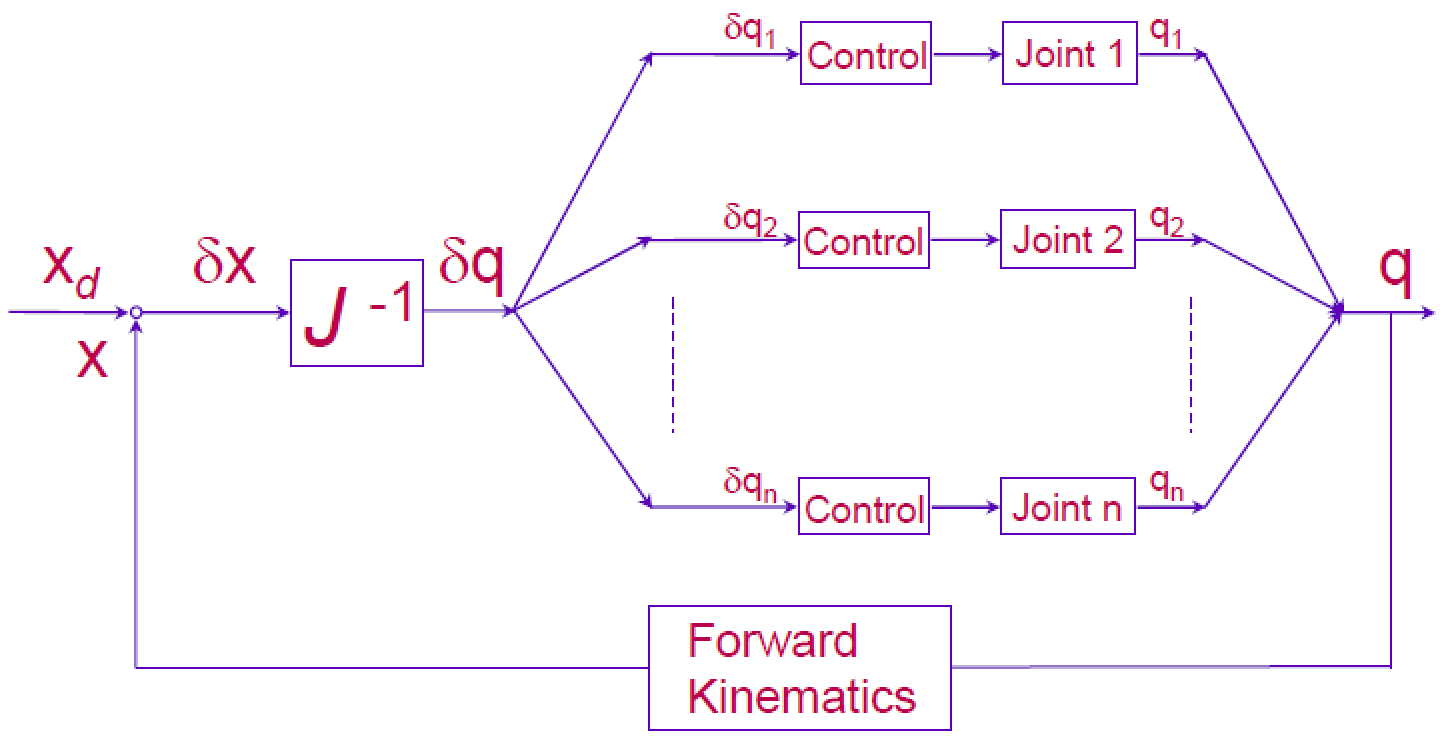
RMRC



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Jacobian Rank

- For RMRC to work J must be invertible
- J is $6 \times n$ and is invertible only if $n = 6$ and full rank
- What can we do if $n > 6$????????

How to calculate inverse Jacobian

$$\zeta = J\dot{q}$$

$$J^T \zeta = J^T J\dot{q}$$

$$\left(J^T J\right)^{-1} J^T \zeta = \left(J^T J\right)^{-1} J^T J\dot{q}$$

$$\left(J^T J\right)^{-1} J^T \zeta = \dot{q}$$

$$\dot{q} = J^+ \zeta$$

$$J^+ = \left(J^T J\right)^{-1} J^T$$

How to calculate J⁺

- Most difficult method (from definition):

$$J^{+} = \left(J^{T} J\right)^{-1} J^{T}$$

- Simplest Method (SVD):

$$J = U \sum V^{T} \qquad \sigma_{ij}^{+} = 1/\sigma_{ij}, \quad \text{when } \sigma_{ij} \neq 0$$

$$J^{+} = U^{T} \sum^{+} V$$

Manipulability

$$\mu = \prod_{i=1}^{m} \sigma_{ii}$$

If the robot is not redundant (n<=6)

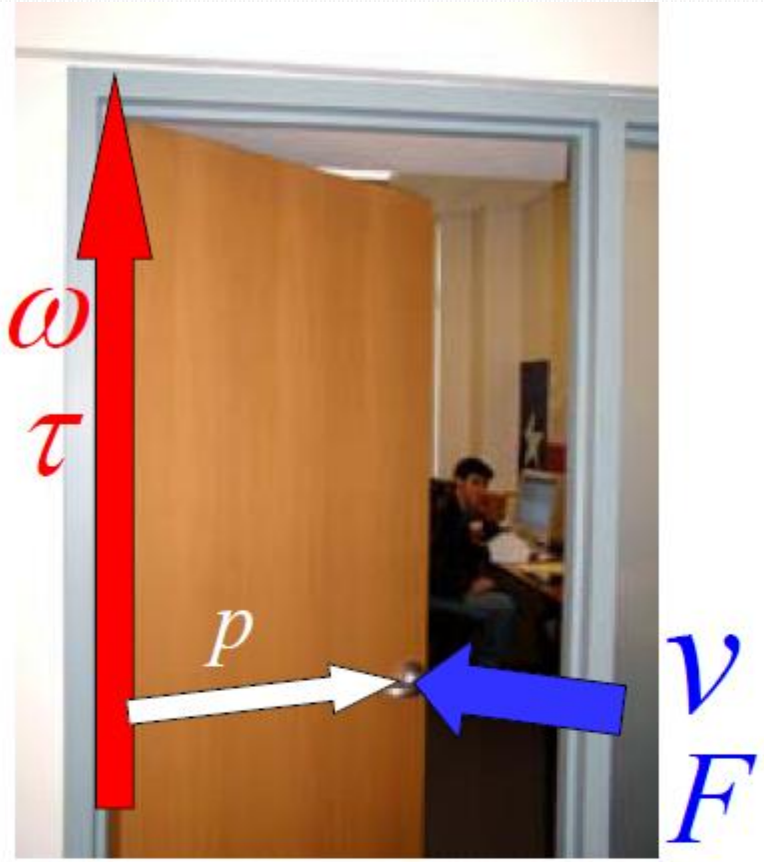
$$\mu = \left| \det \quad J \right|$$

Example (Planar RR)

$$J = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \end{bmatrix}$$

$$\mu = |det\ J| = a_1 a_2 |s_2|$$

Velocity-Force Relation



$$v = \omega \times p$$

$$\tau = p \times F$$

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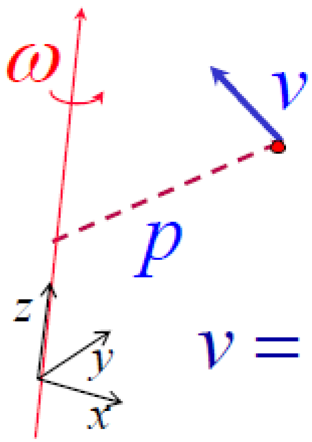
Velocity Force Duality

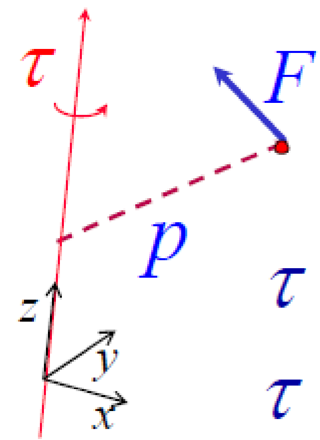


$$v = \omega \times p$$

$$v = -\hat{p}\,\omega$$

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} -p_y \\ p_x \end{pmatrix} \dot{\theta}$$

$$\boxed{v = J\,\dot{\theta}}$$



$$\tau = p \times F$$

$$\tau = \hat{p}\;F$$

$$\tau = (-\hat{p})^T\;F$$

$$\tau = \begin{pmatrix} -p_y & p_x \end{pmatrix} \begin{pmatrix} F_x \\ F_y \end{pmatrix}$$

$$\boxed{\tau = J^T F}$$

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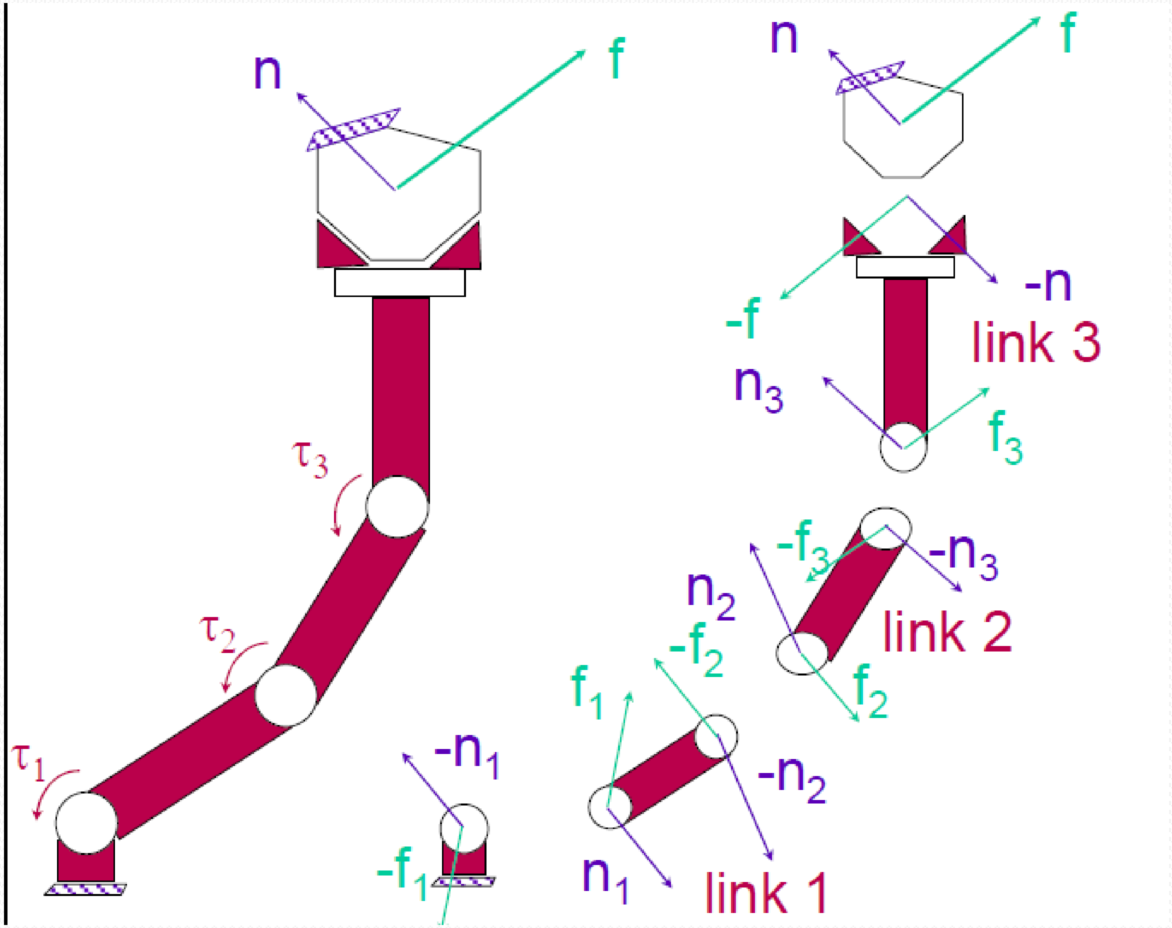
Velocity Force Duality

$$\zeta = J\dot{\theta}$$

$$\tau = J^T F$$

$$F = \begin{bmatrix} f \\ n \end{bmatrix}$$

Statics



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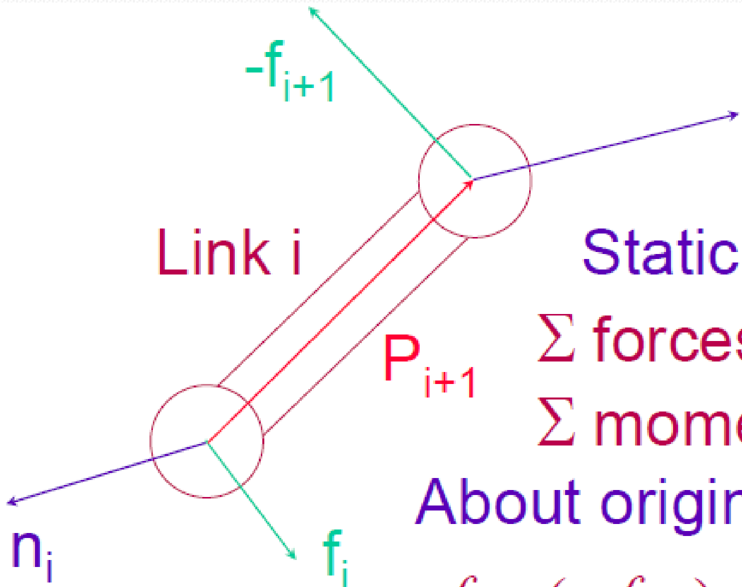
Internal Force Elimination



Static Equilibrium

Σ forces = 0

Σ moments / a point = 0

About origin {i}

$$f_i + (-f_{i+1}) = 0$$

$$n_i + (-n_{i+1}) + P_{i+1} \times (-f_{i+1}) = 0$$

$$f_i = f_{i+1}$$

$$n_i = n_{i+1} + P_{i+1} \times f_{i+1}$$

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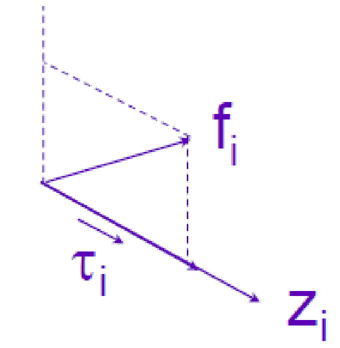
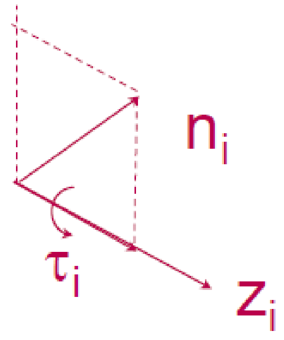
How to do the elimination



Prismatic Joint

$$\tau_i = f_i^T Z_i$$



Revolute Joint

$$\tau_i = n_i^T Z_i$$

Algorithm

$$^{n}f_n = {}^{n}f$$

$$^{n}n_n = {}^{n}n + {}^{n}P_{n+1} \times {}^{n}f$$

$$^{i}f_i = {}^{i}_{i+1}R.\,{}^{i+1}f_{i+1}$$

$$^{i}n_i = {}^{i}_{i+1}R.\,{}^{i+1}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_i$$

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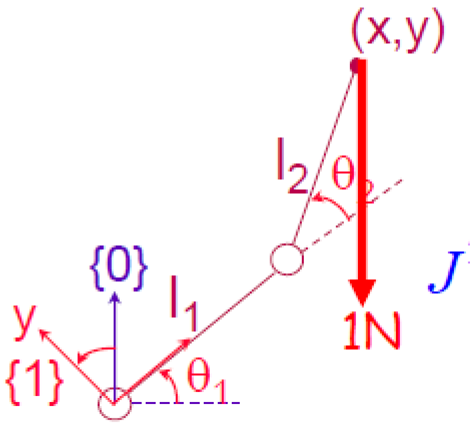
Example



$$J = \begin{pmatrix} -(l_1 S1 + l_2 S12) & -l_2 S12 \\ l_1 C1 + l_2 C12 & l_2 C12 \end{pmatrix}$$

$$J^T = \begin{pmatrix} -(l_1 S1 + l_2 S12) & l_1 C1 + l_2 C12 \\ -l_2 S12 & l_2 C12 \end{pmatrix}$$

$$\boxed{\tau = J^T F}$$

$$l_1 = l_2 = 1; \quad \theta_1 = 0; \; \theta_2 = 60^\circ$$

$$\tau = \begin{pmatrix} -(l_1 S1 + l_2 S12) & l_1 C1 + l_2 C12 \\ -l_2 S12 & l_2 C12 \end{pmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -\begin{bmatrix} l_1 C1 + l_2 C12 \\ l_2 C12 \end{bmatrix} = -\begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}$$

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